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LETTER TO THE EDITOR

Quantum groups constructed from the non-standard braid group representations in the Faddeev–Reshetikhin–Takhtajan approach. II

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Abstract. From the non-standard braid group representation for the fundamental representation of $sl_q(n)$, we obtain the corresponding algebra in the Faddeev–Reshetikhin–Takhtajan approach. The main features of the algebra consist of $(X_k^\pm)^2=0$, $k=1,2,\ldots,n-1$, which obliterates half of the generalized Serre relations; an unusual scalar product normalization of the underlying root vectors, besides the presence of two parameters, one of which is a root of unity. Explicit detail is given for the $n=5$ case. An alternative interpretation suggests that such non-standard solutions are closely connected with the quantum superalgebras.

In a previous paper [1], we have shown that the quantum group [2–8] constructed, in the Faddeev–Reshetikhin–Takhtajan (FRT) approach [5, 6], from the non-standard braid group representation (BGR) for the spin 1 case of $sl_q(2)$ turns out to be the *modular* Hopf algebra [9], namely $sl_\lambda(2)$ at λ being a root of unity and $(X_k^\pm)^{2j+1}=0$.

In this paper, we extend the analysis to the non-standard BGR for the fundamental representation of $sl_q(n)$. We treat the $n=5$ case in detail, and we discuss the general n case.

Our main observations can be stated as follows.

(a) The decoupling phenomenon found in the $sl_\lambda(2)$ case [1] (namely the disappearance of the t parameter in the theory) is only partial for $sl_q(n)$ for $n\geqslant 3$. So the general algebra contains two parameters: $\lambda\equiv\omega^{-1/2}=e^{-\pi i/2}$ (a root of unity) and t. The resulting algebra represents a distortion of the usual $sl_q(n)$.

(b) The basic generators $X_k^\pm$ satisfy $(X_k^\pm)^2=0$, $k=1,2,\ldots,n-1$. Such relations (absent in the standard cases) *obliterate* half of the so-called generalized Serre relations [10] stated by Drinfeld [2] and Jimbo [3],

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_q(X_i^\pm)^{1-a_{ij}-k}(X_j^\pm)(X_i^\pm)^k=0\qquad i\neq j. \tag{1}$$

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where a_{ij} = Cartan matrix element and the square bracket is the q-analogue of the binomial coefficients.

(c) An alternative interpretation is given in terms of the quantum superalgebra.

The FRT formalism [5, 6, 11] determines the algebraic structure of the $L_{ij}^{\pm}$ operators from a given BGR, $\check{R}$, with the following two equations:

$$(L^{\pm}\otimes L^{\pm})\check{R}=\check{R}(L^{\pm}\otimes L^{\pm}) \tag{2}$$

$$(L^{-}\otimes L^{+})\check{R}=\check{R}(L^{+}\otimes L^{-}). \tag{3}$$

Here $L^{\pm}$ are $n\times n$ matrices of the $L_{ij}^{\pm}$ operators and $\check{R}$ is a known $n^2\times n^2$ matrix. A standard $\check{R}$ gives back the known standard algebra; non-standard $\check{R}_{\mathrm{NS}}$, as alternative solutions to BGR, hopefully will generate new algebras [1, 12].

Next we consider the BGR for the fundamental representation of $\mathrm{sl}_q(5)$. As a concrete example, we consider the $n=5$ case, which has sufficiently rich structure to allow an immediate generalization to arbitrary n, yet is still tractable. A simpler case $\mathrm{sl}_q(3)$ which we have also studied would be inadequate to shed light on a subtle point on the different scales involving (a_{ij}). See the discussion following equation (36).

(A) The standard BGR: $\check{R}$ here is a 25×25 matrix which consists of nine block diagonal entries [3, 4]:

$$\check{R}=[A_1, A_2, A_3, A_4, A_5, A_4, A_3, A_2, A_1] \tag{4}$$

where A_m is an $m\times m$ matrix with the following structure:

$$A_1=1 \qquad A_2=\begin{pmatrix}0 & t\\ t & \rho\end{pmatrix} \qquad A_3=\begin{pmatrix}0 & 0 & t\\ 0 & 1 & 0\\ t & 0 & \rho\end{pmatrix} \qquad A_4=\begin{pmatrix}0 & 0 & 0 & t\\ 0 & 0 & t & 0\\ 0 & t & \rho & 0\\ t & 0 & 0 & \rho\end{pmatrix}$$
$$A_5=\begin{pmatrix}0 & 0 & 0 & 0 & t\\ 0 & 0 & 0 & t & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & t & 0 & \rho & 0\\ t & 0 & 0 & 0 & \rho\end{pmatrix} \tag{5}$$

where $\rho\equiv 1-t^2$.

The pattern is clear.

(a) The *even* dimensional A_{2m} matrix has t along the skew-diagonal and ρ along the lower main diagonal. All other entries are zero. (6)

(b) The *odd* dimensional A_{2m+1} matrix has t along the skew-diagonal *except* 1 *at the centre*, and ρ along the lower main diagonal. All other entries are zero. (7)

(B) The non-standard BGR $\check{R}_{\mathrm{NS}}$ here is only slightly distorted from $\check{R}$ [14, 15]:

$$\check{R}_{\mathrm{NS}}=[A_1, A_2, A'_3, A_4, A_5, A_4, A'_3, A_2, A_1]. \tag{8}$$

The difference lies only in the *centre* of those *odd* dimensional blocks A'_{2m+1}. Whereas the standard solution A_{2m+1} contains 1 at the centre, the non-standard solution A'_{2m+1} have the option to replace any (or all) 1 by ωt^2 where

$$\omega = e^{\pi i} \qquad \text{(a root of unity).} \tag{9}$$

Among these various combinations, we find one particular solution $\check{R}_{\rm NS}$ by the following prescription. We replace the 1 by ωt^2 in the two A'_3 while leaving A_5 (as well as the last A_1) untouched. The implications of this choice will be made clear in the discussion following equation (36).

For the spectral parameter-dependent form of $\check{R}_{\rm NS}(x)$ for the fundamental representations of $\mathrm{sl}_q(n)$ (as well as those of B_n, C_n, D_n), the reader is referred to [16].

We next consider the $L^{\pm}_{ij}$ algebra generated from $\check{R}_{\rm NS}$ for the fundamental representation of $\mathrm{sl}_q(5)$. We write $L^+(L^-)$ as 5×5 upper (lower) triangular matrices. Equations (2), (3) and (8) produce a large set of constraints for $L^{\pm}_{ij}$ which are summarized in the appendix. In $L^{\pm}$, the diagonal elements $L^{\pm}_{ii}$ (or a rescaled version thereof, see equation (17)) corresponds to the commuting Cartan subalgebra; the line above (below) the diagonal gives a set of the *linearly independent* generators $\tilde{X}^{\pm}_k$ in the Chevalley basis, $k=1,\ldots,4$. The other off-diagonal elements are *dependent* quantities. The explicit structure for $L^{\pm}$ reads ($\tau \equiv t - t^{-1}$):

$$L^+ = \begin{bmatrix} K_1 & \tau\tilde{X}_1^+ & \tau K_2^{-1}[\tilde{X}_1^+,\tilde{X}_2^+] & \tau K_2^{-1}K_3^{-1}[\tilde{X}_1^+,[\tilde{X}_2^+,\tilde{X}_3^+]] & \tau K_2^{-1}K_3^{-1}K_4^{-1}[\tilde{X}_1^+,[\tilde{X}_2^+,[\tilde{X}_3^+,\tilde{X}_4^+]]] \\ 0 & K_2 & \tau\tilde{X}_2^+ & \tau K_3^{-1}[\tilde{X}_2^+,\tilde{X}_3^+] & \tau K_3^{-1}K_4^{-1}[\tilde{X}_2^+,[\tilde{X}_3^+,\tilde{X}_4^+]] \\ 0 & 0 & K_3 & \tau\tilde{X}_3^+ & \tau K_4^{-1}[\tilde{X}_3^+,\tilde{X}_4^+] \\ 0 & 0 & 0 & K_4 & \tau\tilde{X}_4^+ \\ 0 & 0 & 0 & 0 & K_5 \end{bmatrix} \tag{10}$$

$$L^- = \begin{bmatrix} K_1^{-1} & 0 & 0 & 0 & 0 \\ -\tau\tilde{X}_1^- & K_2^{-1} & 0 & 0 & 0 \\ \tau K_2[\tilde{X}_1^-,\tilde{X}_2^-] & -\tau\tilde{X}_2^- & K_3^{-1} & 0 & 0 \\ -\tau K_2K_3[\tilde{X}_1^-,[\tilde{X}_2^-,\tilde{X}_3^-]] & \tau K_3[\tilde{X}_2^-,\tilde{X}_3^-] & -\tau\tilde{X}_3^- & K_4^{-1} & 0 \\ \tau K_2K_3K_4[\tilde{X}_1^-,[\tilde{X}_2^-,[\tilde{X}_3^-,\tilde{X}_4^-]]] & -\tau K_3K_4[\tilde{X}_2^-,[\tilde{X}_3^-,\tilde{X}_4^-]] & \tau K_4[\tilde{X}_3^-,\tilde{X}_4^-] & -\tau\tilde{X}_4^- & K_5^{-1} \end{bmatrix}. \tag{11}$$

We have denoted $L^+_{ii} \equiv K_i$; $L^-_{ii} = (L^+_{ii})^{-1}$ is a consequence of the constraints. (See appendix, equations (A9), (A10).) We may take four linearly independent (out of five) K by imposing $\Pi^5_{i=1} K_i = 1$. But this is achieved alternatively by a rescaling of the K. (See equation (18).)

In (10), and similarly in (11), there are equivalent expressions for the multiple commutators on account of the Jacobi identity and other constraints such as $[\tilde{X}^+_i, \tilde{X}^+_j] = 0$, for $|i-j|>1$ (see (A3)). Thus, we can also express

$$L^+_{14} = \tau K_2^{-1}K_3^{-1}[[\tilde{X}_1^+,\tilde{X}_2^+],\tilde{X}_3^+] \tag{12}$$

$$L^+_{15} = \tau K_2^{-1}K_3^{-1}K_4^{-1}[[\tilde{X}_1^+,\tilde{X}_2^+],[\tilde{X}_3^+,\tilde{X}_4^+]] \tag{13}$$

$$= \tau K_2^{-1}K_3^{-1}K_4^{-1}[[[\tilde{X}_1^+,\tilde{X}_2^+],\tilde{X}_3^+],\tilde{X}_4^+] \tag{14}$$

$$L^+_{25} = \tau K_3^{-1}K_4^{-1}[[[\tilde{X}_1^+,\tilde{X}_2^+],\tilde{X}_3^+],\tilde{X}_4^+]. \tag{15}$$

Likewise for the corresponding mirror images L^-_{ji}.

It is clear that these multiple commutators have a natural geometrical interpretation in terms of the simple roots of the corresponding Lie algebra. The off-diagonal entries can be visualized to correspond to the collection of *simple* roots plus the other (dependent) *positive* roots in the following array.

$$\text{`}L^{+}\text{'} \rightarrow \begin{bmatrix} 0 & \alpha_1 & \alpha_1+\alpha_2 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \\ & 0 & \alpha_2 & \alpha_2+\alpha_3 & \alpha_2+\alpha_3+\alpha_4 \\ & & 0 & \alpha_3 & \alpha_3+\alpha_4 \\ & & & 0 & \alpha_4 \\ & & & & 0 \end{bmatrix}. \tag{16}$$

The entries for the L^{-} would correspond to the collection of the *negative* roots. The Weyl reflection symmetry is implicit. Many constraints relating the various $L^{\pm}_{ij}$ can be easily understood in terms of such root vectors.

To write the algebra in a form comparable to the standard $\mathrm{sl}_q(n)$ case [2, 3], we find it desirable to rescale our $\tilde{X}^{\pm}$ by some K factors as was done in the standard case by Takhtajan [6]:

$$L^{\pm}_{ii+1} \equiv \pm\tau\tilde{X}^{\pm}_{i} = \pm\tau t^{\pm(n-2)/2n} X^{\pm}_{i}(K_iK_{i+1})^{\pm 1/2} \qquad i=1,\ldots,n-1. \tag{17}$$

Next, let us define

$$\mathcal{K}^2_i \equiv K_iK^{-1}_{i+1} \tag{18}$$

where

$$[K_i, K_j]=0. \tag{19}$$

We find the basic algebraic structure generated by our non-standard BGR of (8) to consist of the following pieces.

(A) Squares of the $X^{\pm}_{k}$ vanish:

$$(X^{\pm}_{k})^2=0 \qquad k=1,\ldots,4. \tag{20}$$

(B) Corresponding to the *diagonal* Cartan matrix part: $\mathcal{K}_i$ on $X^{\pm}_{i}$ has the form of $\mathrm{sl}_\lambda(5)$ with λ being root of unity:

$$\mathcal{K}_iX^{\pm}_{i}\mathcal{K}^{-1}_{i} = \lambda^{\pm 1}X^{\pm}_{i} \qquad i=1,\ldots,4 \tag{21}$$

where

$$\lambda \equiv \omega^{-1/2} = \mathrm{e}^{-\pi i/2} = \text{root of unity}. \tag{22}$$

(C) Corresponding to the *off-diagonal* Cartan matrix part: $\mathcal{K}_j$ on $X^{\pm}_{i}$ has a split alternating structure, $i \neq j$:

$$\mathcal{K}_jX^{\pm}_{i}\mathcal{K}^{-1}_{j} = (\lambda^{-2}t)^{\pm 1/2}X^{\pm}_{i} \qquad i,j=1,2 \text{ and } i,j=3,4 \tag{23}$$

$$= t^{\mp 1/2}X^{\pm}_{i} \qquad i,j=2,3 \tag{24}$$

$$= X^{\pm}_{i} \qquad |i-j|>1. \tag{25}$$

(D) The q-analogue of the generalized $[J^{+}, J^{-}]$:

$$[X^{+}_{i}, X^{-}_{j}] = \delta_{ij}\tau^{-1}(\mathcal{K}^2_i - \mathcal{K}^{-2}_i) \qquad i=1,\ldots,4. \tag{26}$$

We note that one important reason that the rescaled $X^{\mp}_{i}$ variables are preferable to the $\tilde{X}^{\pm}_{i}$ (or $L^{\pm}_{ii+1}$) is to have the Kronecker delta δ_{ij} in (26). Otherwise, the other unscaled variables would satisfy the equivalent but much less transparent form of (A13).

$$\text{(E)} \qquad [X^{\pm}_{i}, X^{\pm}_{j}]=0 \qquad |j-i|>1. \tag{27}$$

We discuss this algebra in the next two sections.

We now consider the implications on the generalized Serre relations. The basic question is whether the algebra generated by the non-standard BGR is *underconstrained* or *underspecified* if half of the generalized Serre relations (1) are being wiped out by (20). Fortunately, the answer is negative. As we shall see, the condition (20) is quite adequate as a replacement for the missing pieces. On the other hand, we can still keep the other half of the generalized Serre relations (28) below or (27). For clarity, we divide our discussion as follows.

(a) Corresponding to the Cartan matrix element $a_{ij}=0$ part (i.e. $|j-i|>1$): we still have that part of (1):

$$\sum_{k=0}^{1}(-1)^k\begin{bmatrix}1\\ k\end{bmatrix}_t (X_i^\pm)^{1-k}X_j^\pm(X_i^\pm)^k=0 \tag{28}$$

which is equivalent to (27).

(b) Corresponding to the Cartan matrix element $a_{ij}=-1$ part (i.e. $|j-i|=1$), the analogue of (1) now takes the form

$$(X_i^\pm)^2X_{i+1}^\pm-(1+\omega)X_i^\pm X_{i+1}^\pm X_i^\pm-X_{i+1}(X_i^\pm)^2=0. \tag{29}$$

The constraints (20) erase the first and the third term, while the second term now has a vanishing coefficient (since $\omega=-1$ for our non-standard BGR). While these parts of the generalized Serre relations become empty, there is no internal inconsistency.

On the other hand, (1) for the standard $\mathrm{sl}_q(n)$ reads

$$(X_i^\pm)^2X_{i+1}^\pm-(q+q^{-1})X_i^\pm X_{i+1}^\pm X_i^\pm+X_{i+1}(X_i^\pm)^2=0. \tag{30}$$

The reconciliation between (29) and (30) lies in the observation that our non-standard BGR generates an algebra corresponding to the standard case of $\mathrm{sl}_q(n)$ at least partially at q being root of unity. So in this case, one expects $q+q^{-1}=0$, or $q^4=1$. So (20) is *not* incompatible with the generalized Serre relations.

The correct interpretation of the q-analogue Serre relations is thus as follows.

(i) In the classical limit $q\to 1$, (30) is equivalent to the statement $[\tilde{X}_i^\pm,[\tilde{X}_i^\pm,\tilde{X}_{i+1}^\pm]]=0$, or $2\alpha_i+\alpha_{i+1}$ is not a root of the underlying Lie algebra.

(ii) In the q-analogue standard case, (30) is equivalent to (read t for q):

$$\tilde{X}_i^\pm[\tilde{X}_i^\pm,\tilde{X}_{i+1}^\pm]=t^2[\tilde{X}_i^\pm,\tilde{X}_{i+1}^\pm]\tilde{X}_i^\pm. \tag{31}$$

(iii) In the non-standard case, we have (see appendix, equations (A10), (A13)), instead, the *outside* anticommutator:

$$\{\tilde{X}_i^\pm,[\tilde{X}_i^\pm,\tilde{X}_{i+1}^\pm]\}=0. \tag{32}$$

Equation (32) is equivalent to (29) by (17). Actually this is reduced to an identity by (20).

We now attempt to give an interpretation of the algebra (21)-(27) in terms of the root vector language. For the standard $\mathrm{sl}_q(n)$ algebra [2, 3], the corresponding equation reads

$$\mathcal{K}_iX_j^\pm\mathcal{K}_i^{-1}=q^{\pm a_{ij}/2}X_j^\pm. \tag{33}$$

With the usual normalization for $\mathrm{sl}(n)$, $a_{ij}=-1+3\delta_{ij}$, $|i-j|\le 1$, we see that for $i=j$, (21) is formally the same as (33) for the standard $\mathrm{sl}_q(n)$ with q being replaced by λ (root of unity) for the non-standard case. This feature is what we have found for the $\mathrm{sl}_\lambda(2)$ case [1]. For $i\ne j$, only (24) and (25) have the form (33) except that our parameter

t in general is not a root of unity. On the other hand, (23) corresponds to a *distorted* value of a_{ij}, away from the canonical value of -1.

In this language, we see that (21)-(24) amount to the following normalization:

$$a_{ii} = 2 \qquad i = 1, \ldots, 4 \tag{34}$$

$$a_{12} = a_{34} = -2 + \ln t/\ln \lambda \tag{35}$$

$$a_{23} = -\ln t/\ln \lambda. \tag{36}$$

Here actually since our $\omega = -1$, λ is essentially pure imaginary i (within a sign), so that $\ln t/\ln \lambda = (2\mathrm{i}/\pi) \ln t$. Equations (35), (36) imply that these scalar products are now alternatingly displaced to $-2 + (2\mathrm{i}/\pi) \ln t$ and to $-(2\mathrm{i}/\pi) \ln t$ (which are symmetrically displaced from their normal value of -1). This alternating feature is precisely related to the way we choose ωt^2 in place of the 1's in the blocks A_{2m+1} (recall that we pick alternating odd blocks in (8)).

This way of choosing our $\check{R}_{\mathrm{NS}}$ implies that all our $X_k^{\pm}$ satisfy (20). Had we chosen a different $\check{R}_{\mathrm{NS}}$, say by modifying the A_5 block into A_5' also, the net effect would be as follows.

(i) Not all $(X_k^{\pm})^2$ would vanish. Instead we would have $(L_{13}^{\pm})^2 = 0$, $(L_{35}^{\pm})^2 = 0$ instead of $(L_{23}^{\pm})^2 = 0$, $(L_{34}^{\pm})^2 = 0$.

(ii) While all the scalar products would have the same value (corresponding to (35)) equation (34) would break down. These features are less appealing than the one we have actually chosen.

The considerations for the $n = 5$ case can be readily generalized to the general n case.

(A) The standard BGR $\check{R}$ is an $n^2 \times n^2$ matrix which consists of $(2n-1)$ blocks of successive (increasing and then decreasing) $m \times m$ A_m matrices whose structure is given in (5)-(7) [3, 4]. For $\mathrm{sl}(n)$, the largest block is $n \times n$:

$$\check{R} = [A_1, A_2, A_3, \ldots, A_{n-1}, A_n, A_{n-1}, \ldots, A_3, A_2, A_1]. \tag{37}$$

(B) The non-standard BGR $\check{R}_{\mathrm{NS}}$ is a slight distortion of $\check{R}$ [13, 14]:

$$\check{R}_{\mathrm{NS}} = [A_1, A_2, A_3', \ldots, A_7', \ldots, A_n, \ldots, A_7', \ldots, A_3', A_2, A_1]. \tag{38}$$

We modify the 1 (to ωt^2) contained in the centre of block $A_{4k+3} \rightarrow A_{4k+3}'$ for $k = 0, 1, \ldots, [n/2]$ where $[f]$ denotes the integer part of the fraction f.

As discussed above, the choice of such a particular non-standard BGR $\check{R}_{\mathrm{NS}}$ will result in the condition that

$$\text{(A)} \qquad (X_k^{\pm})^2 = 0 \qquad k = 1, 2, \ldots, n-1. \tag{39}$$

The rest of the algebra reads

$$\text{(B)} \qquad \mathcal{K}_i X_i^{\pm} \mathcal{K}_i^{-1} = \lambda^{\pm 1} X_i^{\pm} \qquad i = 1, \ldots, n-1 \tag{40}$$

where $\lambda = \mathrm{e}^{-\pi\mathrm{i}/2} =$ root of unity.

$$\text{(C)} \qquad \mathcal{K}_j X_i^{\pm} \mathcal{K}_j^{-1} = (\lambda^{-2} t)^{\pm 1/2} X_i^{\pm} \qquad i, j = 2s-1, 2s \tag{41}$$

$$= t^{\mp 1/2} X_i^{\pm} \qquad i, j = 2s, 2s+1 \quad i \neq j \quad s = 1, \ldots, \left[\frac{n-1}{2}\right]. \tag{42}$$

$$\text{(D)} \qquad [X_i^+, X_j^-] = \delta_{ij}\tau^{-1}(\mathcal{K}_i^2 - \mathcal{K}_i^{-2}). \tag{43}$$

$$\text{(E)} \qquad [X_i^\pm, X_j^\pm] = 0 \qquad |j-i|>1. \tag{44}$$

As discussed above, equation (39) makes half of the generalized Serre relations (1) disappear from this algebra, while we still keep the other half in the form of (28) or (27).

The interpretation of (41), (42) is an obvious generalization of that given above for the $n=5$ case.

In light of recent developments, quantum groups associated with the non-standard BGR have been studied by several other authors [16, 17]. For the spin-$\frac{1}{2}$ $\mathrm{sl}_q(2)$ representation, Jing *et al* [12] discussed the resulting new algebra from the conventional approach. On the other hand, Liao and Song [17] discussed this from the viewpoint of the graded algebras, leading to their identification of $\mathrm{sl}_q(1|1)$.

In this letter, our treatment for $\mathrm{sl}_q(n)$ may be regraded as a generalization of that of Jing *et al* [12]. In the conventional approach, we find a new algebra. On the other hand, it is possible to give an alternate interpretation in the graded algebra in the spirit of Liao and Song [17]. Viewed in this light, our algebra for the rank-4 case can be recast in a form which can be identified as that of $\mathrm{sl}_q(3|2)$. The basic ideas are as follows.

(a) The non-standard BGR are obtained from the standard BGR by the replacement of $q \to -q^{-1}$ in certain strategic places. (See e.g. (8).) By introducing a phase factor (or a metric tensor) η (which is a diagonal matrix with 1, except -1 at certain places), we can define a graded Yang–Baxter equation [18].

(b) In our $\mathrm{sl}_q(5)$ case (8) where we have replaced q by $-q^{-1}$ (or in our notation $1 \to \omega t^2, t \equiv q^{-1}$) in two places, the 25×25 η-matrix has -1 in the 5th and the 21st places. The η-matrix will effectively compensate for the -1 (or ω) factor in the $\check{R}$ matrix. Thus in (21), the λ-factor is effectively absent. The vanishing of the square of X_k in (20) is natural in the superalgebraic interpretation. Introduce three sets of boson operators (b) and two sets of fermion operators (a) such that our X_k^+ are constructed from the products $b_1^+a_1, b_2^+a_1, b_2^+a_2, b_3^+a_2$, and our commuting K_i operators are constructed from the alternating $b_r^+b_r$ and $a_s^+a_s$. Equation (21) with λ gone implies that the difference of the number of bosons minus the fermions is a constant. Equation (23) with λ gone becomes the bosonic part of $\mathrm{sl}_q(3|2)$, and (24) becomes the fermionic part of $\mathrm{sl}_q(3|2)$. Equation (26) becomes anticommutators. Equations (25) and (27) remain the same. Details will be given elsewhere [19].

From the non-standard BGR $\check{R}_{\mathrm{NS}}$ for the fundamental representation of $\mathrm{sl}_q(n)$, we have obtained the resulting algebra implied in the FRT approach. We have given a general treatment for the arbitrary n case, after having examined the $n=5$ case in considerable detail. Unlike the simple $\mathrm{sl}_q(2)$ case [1, 12], we encounter an (apparently new) algebra given by (38)–(44) which is a deformation of the standard algebra. We have attempted to give an interpretation in terms of the different scalar product normalization of the corresponding root vectors involved. We have also discussed the implications for the generalized Serre relations.

The question of implementing this algebra into a Hopf algebra will be pursued elsewhere.

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Appendix. Algebraic detail for $n=5$

We summarize here the results from equations (2), (3) and (8). The block diagonal structure of the 25×25 $\check{R}_{\mathrm{NS}}$ of (8) is written in accordance with a specific labelling scheme. Thus, to bring the direct products $L^{\pm}\otimes L^{\pm}$ to conform to the same labelling scheme, certain permutations among the rows and columns are required. Explicitly, we rearrange the columns of the direct product in the following sequence: (1), (2, 6), (3, 7, 11), (4, 8, 12, 16), (5, 9, 13, 17, 21), (10, 14, 18, 22), (15, 19, 23), (20, 24), (25). In other words, the $m\times m$ block collects the $[m, m+n-1, \ldots, m+(n-1)(m-1)]$th columns $(1\leqslant m\leqslant n)$ on the way up, then the $m\times m$ block collects the $[kn, kn+(n-1), \ldots, kn+(n-1)(m-1)]$th columns $(2\leqslant k\leqslant n-1, 1\leqslant m\leqslant n-1)$ on the way down. Likewise for the reshufflings of the rows.

We have (repeated indices are *not* summed):

(1) Diagonal elements commute:

$$[L^{\pm}_{ii}, L^{\pm}_{jj}]=0 \qquad i,j=1,\ldots,n \quad \text{all } \pm \text{ combinations.} \tag{A1}$$

(2) Other vanishing commutators:

(a) Between the diagonal and the off-diagonal elements:

$$\begin{aligned} &[L^{\pm}_{ii}, L^{+}_{jk}]=0 \qquad [L^{\pm}_{ii}, L^{-}_{kj}]=0 \\ &[L^{\pm}_{jj}, L^{+}_{ik}]=0 \qquad [L^{\pm}_{jj}, L^{-}_{ki}]=0 \qquad i<j<k\leqslant n. \\ &[L^{\pm}_{kk}, L^{+}_{ij}]=0 \qquad [L^{\pm}_{kk}, L^{-}_{ji}]=0 \end{aligned} \tag{A2}$$

(b) Among the off-diagonal elements (no indices in common):

$$\begin{aligned} &[L^{+}_{ij}, L^{+}_{kl}]=0 \qquad [L^{-}_{ji}, L^{-}_{lk}]=0 \\ &[L^{+}_{ij}, L^{-}_{lk}]=0 \qquad [L^{-}_{ji}, L^{+}_{kl}]=0 \\ &[L^{+}_{il}, L^{+}_{jk}]=0 \qquad [L^{-}_{li}, L^{-}_{kj}]=0 \qquad i<j<k<l\leqslant n. \\ &[L^{+}_{il}, L^{-}_{kj}]=0 \qquad [L^{-}_{li}, L^{+}_{jk}]=0 \end{aligned} \tag{A3}$$

(3) Non-vanishing commutators $(\tau\equiv t-t^{-1})$:

(a) No indices in common:

$$\begin{aligned} &[L^{+}_{ik}, L^{+}_{jl}]=\tau L^{+}_{il}L^{+}_{jk} \qquad \cdot[L^{-}_{ki}, L^{-}_{lj}]=\tau L^{-}_{li}L^{-}_{kj} \\ &[L^{+}_{jl}, L^{-}_{ki}]=\tau L^{+}_{kl}L^{-}_{ji} \qquad [L^{-}_{lj}, L^{+}_{ik}]=\tau L^{-}_{lk}L^{+}_{ij} \qquad i<j<k<l\leqslant n. \end{aligned} \tag{A4}$$

(b) One common index:

$$\begin{aligned} &[L^{+}_{ij}, L^{+}_{jk}]=\tau L^{+}_{jj}L^{+}_{ik} \qquad [L^{-}_{ji}, L^{-}_{kj}]=\tau L^{-}_{jj}L^{-}_{ki} \\ &[L^{+}_{jk}, L^{-}_{ki}]=\tau L^{+}_{kk}L^{-}_{ji} \qquad [L^{-}_{kj}, L^{+}_{ik}]=\tau L^{-}_{kk}L^{+}_{ij} \qquad i<j<k\leqslant n. \\ &[L^{+}_{ik}, L^{-}_{ji}]=\tau L^{-}_{ii}L^{+}_{jk} \qquad [L^{-}_{ki}, L^{+}_{ij}]=\tau L^{+}_{ii}L^{-}_{kj} \end{aligned} \tag{A5}$$

(c) Two common indices:

$$[L^{+}_{ij}, L^{-}_{ji}]=-\tau[L^{+}_{ii}L^{-}_{jj}-L^{+}_{jj}L^{-}_{ii}] \qquad i<j\leqslant n. \tag{A6}$$

Equation (A6) is the origin of (26) for $j=i+1$.

Equation (A5) gives the entries $L_{ij}^{\pm}$ for $j \geq i+2$ as commutators and iterated multiple commutators in (10) and (11).

(4) Vanishing squares:

$$(L_{i\,i+1}^{+})^2 = 0 \qquad (L_{i+1\,i}^{-})^2 = 0 \qquad i = 1, \ldots, n. \tag{A7}$$

Equation (A7) gives (20). These are the independent requirements as a consequence of the suitably chosen $\check{R}_{\mathrm{NS}}$ of (8). Furthermore,

$$(L_{i\,i+3}^{+})^2 = 0 \qquad (L_{i+3\,i}^{-})^2 = 0 \qquad i \leq n-3. \tag{A8}$$

Unlike (A7), however, (A8) are not independent relations. They actually are identities which can be seen to follow from (A7), the first equation of (A5) and (A12).

(5) q-analogue commutators:

(a) Between the diagonal and the off-diagonal:

$$L_{ii}^{\pm} L_{ij}^{+} L_{ii}^{\pm -1} = c_{ij}^{\pm 1} L_{ij}^{+} \qquad L_{ii}^{\pm} L_{ji}^{-} L_{ii}^{\pm -1} = c_{ij}^{\mp 1} L_{ji}^{-}$$

where

$$c_{ij} = \begin{cases} t & i = \text{odd} \qquad i < j \leq n \\ (\omega t)^{-1} & i = \text{even} \qquad i < j \leq n \end{cases} \tag{A9}$$

$$L_{jj}^{\pm} L_{ij}^{+} L_{jj}^{\pm -1} = d_{ij}^{\pm 1} L_{ij}^{+} \qquad L_{jj}^{\pm} L_{ji}^{-} L_{jj}^{\pm -1} = d_{ij}^{\mp 1} L_{ji}^{-}$$

where

$$d_{ij} = \begin{cases} \omega t & j = \text{even} \qquad i < j \leq n \\ t^{-1} & j = \text{odd} \qquad i < j \leq n. \end{cases} \tag{A10}$$

(b) Among the off-diagonals:

$$L_{ij}^{+} L_{ik}^{+} = r_{ijk} L_{ik}^{+} L_{ij}^{+} \qquad L_{ji}^{-} L_{ki}^{-} = r_{ijk} L_{ki}^{-} L_{ji}^{-}$$

where

$$r_{ijk} = \begin{cases} t & i = \text{odd} \qquad i < j < k \leq n \\ (\omega t)^{-1} & i = \text{even} \qquad i < j < k \leq n \end{cases} \tag{A11}$$

$$L_{ik}^{+} L_{jk}^{+} = s_{ijk} L_{jk}^{+} L_{ik}^{+} \qquad L_{ki}^{-} L_{kj}^{-} = s_{ijk} L_{kj}^{-} L_{ki}^{-}$$

where

$$s_{ijk} = \begin{cases} t & k = \text{odd} \qquad i < j < k \leq n \\ (\omega t)^{-1} & k = \text{even} \qquad i < j < k \leq n \end{cases} \tag{A12}$$

$$L_{ij}^{+} L_{j+1\,j}^{-} = p_{ij} L_{j+1\,j}^{-} L_{ij}^{+} \qquad L_{ji}^{-} L_{jj+1}^{+} = p_{ij} L_{jj+1}^{+} L_{ji}^{-}$$

where

$$p_{ij} = \begin{cases} \omega t & j = \text{even} \qquad i < j \leq n-1 \\ t^{-1} & j = \text{odd} \qquad i < j \leq n-1. \end{cases} \tag{A13}$$

Equations (21)–(26) follow from (A9), (A10) and (A2). It is interesting to note that (A9)–(A13) give the commutators *and* anticommutators in the limit $t \to 1$, $\omega \equiv -1$.

It can be verified that all the constraints stated here become identities as a consequence of the basis algebra given in (20)–(27).

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